Chapter 19 Integer Linear Programming

An Introduction to Optimization

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Wei-Ta Chu

Introduction

- *Integer linear programming* (ILP), or simply *integer programming*, is linear problems with the additional constraint that the solution components be integers.
- Notation
  - The set of integers: $\mathbb{Z}$
  - The set of vectors with *n* integer components: $\mathbb{Z}^n$
  - The set of *m* by *n* matrices with integer entries: $\mathbb{Z}^{m\times n}$
- Express an ILP problem in the following form:

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}$$

Unimodular Matrices

- Definition 19.1. An *m* by *n* integer matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$, $m \leq n$, is *unimodular* if all its nonzero *m*th-order minors are ± 1 .
- Consider the linear equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$, $m \leq n$ Let $\boldsymbol{B}$ be a corresponding basis matrix (an *m* by *m* matrix consisting of *m* linearly independent columns of $\boldsymbol{A}$). Then, the unimodularity of $\boldsymbol{A}$ is equivalent to $|\det \boldsymbol{B}| = 1$ for any such $\boldsymbol{B}$.

- A ***pth-order minor*** of an $m \times n$ matrix *A*, with $p \leq \min\{m, n\}$, is the determinant of a $p \times p$ matrix obtained from *A* by deleting *m-p* rows and *n-p* columns

Unimodular Matrices

- Lemma 19.1. Consider the linear equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} \in \mathbb{Z}^{m\times n}$, $m \le n$, is unimodular and $\boldsymbol{b} \in \mathbb{Z}^m$. Then, all basic solutions have integer components.
- Proof. Suppose that the first *m* columns of $\boldsymbol{A}$ constitute a basis, and that $\boldsymbol{B}$ is the invertible *m* by *m* matrix composed of these columns. Then the corresponding basic solution is

$$\boldsymbol{x}^* = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

  Because all the elements of $\boldsymbol{A}$ are integers, $\boldsymbol{B}$ is an integer matrix. Moreover, because $\boldsymbol{A}$ is unimodular, $|\det \boldsymbol{B}| = 1$. This implies that the inverse $\boldsymbol{B}^{-1}$ is also an integer matrix. Therefore, $\boldsymbol{x}^*$ is an integer vector.

Unimodular Matrices

- Corollary 19.1. Consider the LP constraint $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, where $\boldsymbol{A}$ is unimodular, $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$, $m \leq n$, and $\boldsymbol{b} \in \mathbb{Z}^m$. Then, all basic feasible solutions have integer components.
- Unimodularity allows us to solve ILP problems using the simplex method. Consider the ILP problem

$$\begin{aligned} &\text{minimize } \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0} \quad \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$, $m \leq n$, $\boldsymbol{b} \in \mathbb{Z}^m$. The corollary tells us that if we consider the associated LP problem

$$\begin{aligned} &\text{minimize } \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

the optimal basic feasible solution is an integer vector.

Example

- Consider the following ILP problem

$$
\begin{aligned}
\text{maximize } & 2x_1 + 5x_2 \\
\text{subject to } & x_1 + x_3 = 4 \\
& x_2 + x_4 = 6 \\
& x_1 + x_2 + x_5 = 8 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0 \\
& x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}
\end{aligned}
$$

We can write this problem in matrix form with

$$
\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}
$$

It is easy to check that $\boldsymbol{A}$ is unimodular.

Example

- Hence, the ILP problem above can be solved by solving the LP problem

$$\begin{aligned} \text{maximize } & 2x_1 + 5x_2 \\ \text{subject to } & x_1 + x_3 = 4 \\ & x_2 + x_4 = 6 \\ & x_1 + x_2 + x_5 = 8 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

- This was done in Example 16.2 using the simplex method, yielding optimal solution $[2, 6, 2, 0, 0]^T$

Unimodular Matrices

- In general, when the matrix $\boldsymbol{A}$ is not unimodular, the simplex method applied to the associated LP problem yields a noninteger optimal solution. But there is some exception.
- Suppose that $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$, $m \leq n$, and $\boldsymbol{b} \in \mathbb{Z}^{m}$, as long as each *m* by *m* basis matrix $\boldsymbol{B}$ consisting of columns of $\boldsymbol{A}$ corresponding to a basic feasible solution has the property that $|\det \boldsymbol{B}| = 1$, we can use the argument in the proof of Lemma 19.1 to conclude that the basic feasible solution is an integer vector.
- Equivalently, we can draw this conclusion if each basis submatrix $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $|\det \boldsymbol{B}| \neq 1$ corresponds to a *nonfeasible* basic solution.

Example

- Consider the ILP problem

$$\begin{aligned} \text{minimize} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + x_2 + x_4 = 3 \\ & x_1 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \\ & x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z} \end{aligned}$$

  Can this ILP problem be solved using the simplex method?

- The matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  is not unimodular.

Example

- Indeed, this matrix has one (and only one) basis submatrix with determinant other than ± 1 , consisting of the first, fourth, and fifth columns of $\boldsymbol{A}$.
- Indeed, if we write $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_4, \boldsymbol{a}_5]$, then $\det \boldsymbol{B} = -2$. However, a closer examination of this matrix an the vector $\boldsymbol{b} = [2, 3, 3]^T$ reveals that the corresponding basic solution is not feasible: $\boldsymbol{B}^{-1}\boldsymbol{b} = [-1, 2, 4]^T$ (which, coincidentally, happens to be an integer vector). Therefore, for this problem, applying the simplex method will produce an integer optimal basic feasible solution.

Example

- We begin by forming the first tableau,

$$
\begin{array}{ccccccc}
 & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\
 & -2 & 1 & 1 & 0 & 0 & 2 \\
 & -1 & 1 & 0 & 1 & 0 & 3 \\
 & 1 & 0 & 0 & 0 & 1 & 3 \\
\boldsymbol{c}^T & -1 & -2 & 0 & 0 & 0 & 0
\end{array}
$$

We have $r_2 = -2$. Therefore, we introduce $\boldsymbol{a}_2$ into the new basis. We calculate the ratios $y_{i0}/y_{i2}, y_{i2} > 0$, to determine the pivot element:

$$
\frac{y_{10}}{y_{12}} = \frac{2}{1} \qquad \frac{y_{20}}{y_{22}} = \frac{3}{1}
$$

We will use y_{12} as the pivot.

Example

- Performing elementary row operations, we obtain the second tableau,

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
	-2	1	1	0	0	2
	1	0	-1	1	0	1
	1	0	0	0	1	3
$\boldsymbol{r}^T$	-5	0	2	0	0	4

We now have $r_1 = -5 < 0$. Therefore, we introduce $\boldsymbol{a}_1$ into the new basis. We next calculate $y_{i0}/y_{i2}, y_{i2} > 0$ to determine the pivot element:

$$\frac{y_{20}}{y_{21}} = \frac{1}{1} \qquad \frac{y_{30}}{y_{31}} = \frac{3}{1}$$

We will use y_{21} as the pivot.

Example

- Performing row elementary operations, we obtain the third tableau,

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
	0	1	−1	2	0	4
	1	0	−1	1	0	1
	0	0	1	1	1	2
$\boldsymbol{r}^T$	0	0	−3	5	0	9

We have $r_3 = -3 < 0$. Therefore, we introduce $\boldsymbol{a}_3$ into the new basis. We next calculate the ratios $y_{i0}/y_{i2}, y_{i2} > 0$ to determine the pivot element,

$$\frac{y_{30}}{y_{33}} = \frac{2}{1}$$

We will use y_{33} as the pivot.

Example

- Performing row elementary operations, we obtain the fourth tableau,

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
	0	1	0	1	1	6
	1	0	0	0	1	3
	0	0	1	-1	1	2
$\boldsymbol{r}^T$	0	0	0	2	3	15

All reduced cost coefficients are now positive, which means that the current solution is optimal. This solution is $[3, 6, 2, 0, 0]^T$

Unimodular Matrices

- We next consider ILP problems of the form

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}$$

- We have seen in Section 15.5 that we can transform the inequality constraint into standard form by introducing slack variables. Doing so would lead to a new problem in standard form for which the constraint has the form $[\boldsymbol{A}, \boldsymbol{I}]\boldsymbol{y} = \boldsymbol{b}$ (where the vector $\boldsymbol{y}$ contains $\boldsymbol{x}$ and the slack variables). To deal with matrices of the form $[\boldsymbol{A}, \boldsymbol{I}]$, we need another definition.

Unimodular Matrices

- Definition 19.2. An *m* by *n* integer matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$ is *totally unimodular* if all its nonzero minors are ± 1
- By minors here we mean *p*th-rder minors for $p \leq \min(m, n)$ Equivalently, a matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$ is totally unimodular if and only if all its square invertible submatrices have determinant ± 1 . By a *submatrix* of $\boldsymbol{A}$ we mean a matrix obtained by removing some columns and rows of $\boldsymbol{A}$.
- If an integer matrix is totally unimodular, then each entry is 0, 1, or, -1.

Proposition 19.1

- Proposition 19.1. If an *m* by *n* integer matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$ is totally unimodular, then the matrix $[\boldsymbol{A}, \boldsymbol{I}]$ is unimodular.
- Proof. Let $\boldsymbol{A}$ satisfy the assumptions of the proposition. We will show that any *m* by *m* invertible submatrix of $[\boldsymbol{A}, \boldsymbol{I}]$ has determinant ± 1 . We first note that any *m* by *m* invertible submatrix of $[\boldsymbol{A}, \boldsymbol{I}]$ that consists only of columns of $\boldsymbol{A}$ has determinant ± 1 because $\boldsymbol{A}$ is totally unimodular. Moreover, the *m* by *m* submatrix $\boldsymbol{I}$ satisfies $\det \boldsymbol{I} = 1$

Proposition 19.1

- Consider now an *m* by *m* invertible submatrix of $[\boldsymbol{A}, \boldsymbol{I}]$ composed of *k* columns of $\boldsymbol{A}$ and *m*-*k* columns of $\boldsymbol{I}$. Without loss of generality, suppose that this submatrix is composed of the last *k* columns of $\boldsymbol{A}$ and the first *m*-*k* columns of $\boldsymbol{I}$; that is, the *m* by *m* invertible submatrix is

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{a}_{n-k+1} & \cdots & \boldsymbol{a}_n & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_{m-k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{m-k,k} & \boldsymbol{I}_{m-k} \\ \boldsymbol{B}_{k,k} & \boldsymbol{O} \end{bmatrix}$$

where $\boldsymbol{e}_i$ is the *i*th column of the identity matrix. This choice of columns is without loss of generality because we can exchange rows and columns to arrive at this form, and each exchange only changes the sign of the determinant.

Proposition 19.1

- Moreover, note that $\det \boldsymbol{B} = \pm \det \boldsymbol{B}_{k,k}$. Thus, $\boldsymbol{B}_{k,k}$ is invertible because $\boldsymbol{B}$ is invertible. Moreover, because $\boldsymbol{B}_{k,k}$ is a submatrix of $\boldsymbol{A}$ and $\boldsymbol{A}$ is totally unimodular, $\det \boldsymbol{B}_{k,k} = \pm 1$ Hence, $\det \boldsymbol{B} = \pm 1$ also. Thus any *m* by *m* invertible submatrix $[\boldsymbol{A}, \boldsymbol{I}]$ has determinant ± 1 , which implies that $[\boldsymbol{A}, \boldsymbol{I}]$ is unimodular.

Unimodular Matrices

- Corollary 19.2. Consider the LP constraint

$$[\boldsymbol{A}, \boldsymbol{I}]\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

where $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$ is totally unimodular and $\boldsymbol{b} \in \mathbb{Z}^m$. Then, all basic feasible solutions have integer components.

- Total unimodularity of $\boldsymbol{A}$ allows us to solve ILP problems of the following form using the simplex method:

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \qquad \boldsymbol{b} \in \mathbb{Z}^m \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}$$

Unimodular Matrices

- Specifically, we first consider the associated LP problem

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

- If $\boldsymbol{A}$ is totally unimodular, then the corollary above tells us that once we convert this problem into standard form by introducing a slack-variable vector $\boldsymbol{z}$

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to } & [\boldsymbol{A}, \boldsymbol{I}] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{z} \ge \boldsymbol{0} \end{aligned}$$

the optimal basic feasible solution is an integer vector.

Unimodular Matrices

- This means that we can apply the simplex method to the LP problem above to obtain a solution to the original ILP problem.
- Note that although we only needed the $\boldsymbol{x}$ part of the solution to be integer, the slack-variable vector $\boldsymbol{z}$ is automatically integer for any integer $\boldsymbol{x}$, because both $\boldsymbol{A}$ and $\boldsymbol{b}$ only contain integers.

Example

- Consider the following ILP problem:

$$\begin{aligned} \text{maximize } & 2x_1 + 5x_2 \\ \text{subject to } & x_1 \leq 4 \\ & x_2 \leq 6 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

- This problem can be written in the matrix form above with

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

Example

- It is easy to check that $\boldsymbol{A}$ is totally unimodular. Hence, the ILP problem above can be solved by soling the LP problem

$$
\begin{aligned}
\text{maximize } & 2x_1 + 5x_2 \\
\text{subject to } & x_1 + x_3 = 4 \\
& x_2 + x_4 = 6 \\
& x_1 + x_2 + x_5 = 8 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Example

- Consider the following ILP problem:

$$\begin{aligned} \text{maximize } & x_1 + 2x_2 \\ \text{subject to } & -2x_1 + x_2 \leq 2 \\ & x_1 - x_2 \geq -3 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \qquad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

- We first express the given problem in the equivalent form:

$$\begin{aligned} \text{minimize } & -x_1 - 2x_2 \\ \text{subject to } & -2x_1 + x_2 \leq 2 \\ & -x_1 + x_2 \leq 3 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \qquad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Example

- We next represent the problem above in standard form by introducing slack variables x_3, x_4, x_5 to obtain

$$\begin{aligned} \text{minimize} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + x_2 + x_4 = 3 \\ & x_1 + x_5 = 3 \\ & x_i \geq 0, i = 1, ..., 5 \end{aligned}$$

- The problem is now of the form in Example 19.2, where the simplex method was used. Recall that the solution is $[3, 6, 2, 0, 0]^T$. Thus, the solution to the original problem is $\boldsymbol{x}^* = [3, 6]^T$

Example

- Note that the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$$

is not totally unimodular, because it has an entry (-2) not equal to 0, 1, or, -1. Indeed, the matrix $[\boldsymbol{A}, \boldsymbol{I}]$ is not unimodular. However, in this case, the simplex method still produces an optimal solution to the ILP.

The Gomory Cutting-Plane Method

- In 1958, R.E. Gomory proposed a method where noninteger optimal solutions obtained using the simplex method are successively removed from the feasible set by adding constraints that exclude these noninteger solutions from the feasible set.
- The additional constraints, referred to as *Gomory cuts*, do not eliminate integer feasible solutions from the feasible set. The process is repeated until the optimal solution is an integer vector.

The Gomory Cutting-Plane Method

- Definition 19.3. The floor of a real number, denoted $\lfloor x \rfloor$, is the integer obtained by rounding x toward $-\infty$.
- Consider the ILP problem

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}$$

We begin by applying the simplex method to obtain an optimal basic feasible solution to the LP problem

$$\begin{aligned} \text{minimize } & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

The Gomory Cutting-Plane Method

- As usual, suppose that the first m columns form the basis for the optimal basic feasible solution. The corresponding canonical augmented matrix is

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_i & \cdots & \boldsymbol{a}_m & \boldsymbol{a}_{m+1} & \cdots & \boldsymbol{a}_n & \boldsymbol{y}_0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & y_{1,m+1} & \cdots & y_{1,n} & y_{10} \\ 0 & 1 & \cdots & 0 & \cdots & 0 & y_{2,m+1} & \cdots & y_{2,n} & y_{20} \\ \vdots & \vdots & & \vdots & & \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & y_{i,m+1} & \cdots & y_{i,n} & y_{i0} \\ \vdots & \vdots & & \vdots & & \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & y_{m,m+1} & \cdots & y_{m,n} & y_{m0} \end{bmatrix}$$

- Consider the *i*th component of the optimal basic feasible solution, y_{i0}. Suppose that y_{i0} is not an integer.

The Gomory Cutting-Plane Method

- Note that any feasible vector $\boldsymbol{x}$ satisfies the equality constraint (taken from the ith row)

  $$x_i + \sum_{j=m+1}^{n} y_{ij} x_j = y_{i0}$$

  We use this equation to derive an additional constraint that would eliminate the current optimal noninteger solution from the feasible set without eliminating any integer feasible solution. To see how, consider the inequality constraint
  $$x_i + \sum_{j=m+1}^{n} \lfloor y_{ij} \rfloor x_j \le y_{i0}$$

  Because $\lfloor y_{ij} \rfloor \le y_{ij}$, any $\boldsymbol{x} \ge \mathbf{0}$ that satisfies the first equality constraint above also satisfies this inequality constraint. Thus, any feasible $\boldsymbol{x}$ satisfies this inequality constraint.

The Gomory Cutting-Plane Method

- Moreover, for any integer feasible vector $\boldsymbol{x}$, the left-hand side of the inequality constraint is an integer. Therefore, any integer feasible vector $\boldsymbol{x}$ also satisfies

$$x_i + \sum_{j=m+1}^{n} \lfloor y_{ij} \rfloor x_j \leq \lfloor y_{i0} \rfloor$$

- Subtracting this inequality from the equation above, we deduce that any integer feasible vector satisfies

$$\sum_{j=m+1}^{n} (y_{ij} - \lfloor y_{ij} \rfloor) x_j \geq y_{i0} - \lfloor y_{i0} \rfloor$$

The Gomory Cutting-Plane Method

- Next, notice that the optimal basic feasible solution above does not satisfy this inequality, because the left-hand side for the optimal basic feasible solution is 0, but the right-hand side is a positive number.
- Therefore, if we impose the additional inequality constraint above to the original LP problem, the new constraint set would be such that the current optimal basic feasible solution is no longer feasible, but yet every *integer* feasible vector remains feasible. This new constraint is called a *Gomory cut*.

The Gomory Cutting-Plane Method

- To transform the new LP problem into standard form, we introduce the surplus variable x_{n+1} to obtain the equality constraint

  $$\sum_{j=m+1}^{n}(y_{ij} - \lfloor y_{ij} \rfloor)x_j - x_{n+1} = y_{i0} - \lfloor y_{i0} \rfloor$$

  For convenience, we will also call this equality constraint a *Gomory cut*. By augmenting this equation into $\boldsymbol{A}$ and $\boldsymbol{b}$ or canonical versions of them (e.g., in the form of a simplex tableau), we obtain a new LP problem in standard form.
- We can then solve the new problem using the simplex method and examine the resulting optimal basic feasible solution.

The Gomory Cutting-Plane Method

- If the solution satisfies the integer constraints, then we are done.
- If the solution does not satisfy the integer constraints, we introduce another Gomory cut and repeat the process. We call this procedure the *Gomory cutting-plane method*.
- Note that in applying this method, we only need to introduce enough cuts to satisfy the integer constraints for the original ILP problem. The additional variables introduced by slack variables or by the Gomory cuts are not constrained to be integers.

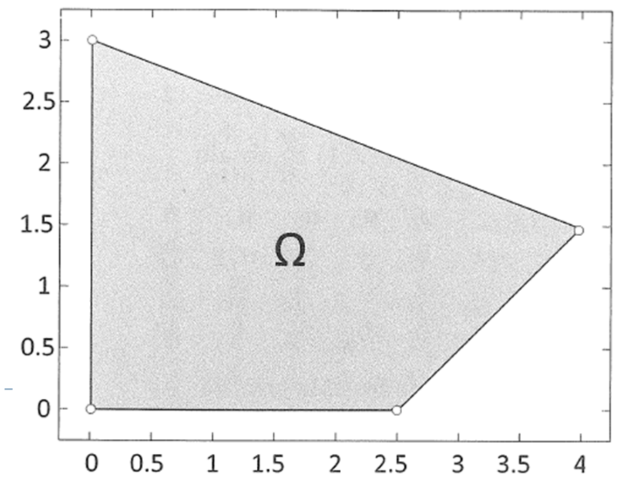
Example

- Consider the following ILP problem:

$$\begin{aligned} &\text{maximize } 3x_1 + 4x_2 \\ &\text{subject to } \tfrac{2}{5}x_1 + x_2 \le 3 \qquad x_1, x_2 \ge 0 \\ &\qquad\qquad \tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 \le 1 \qquad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

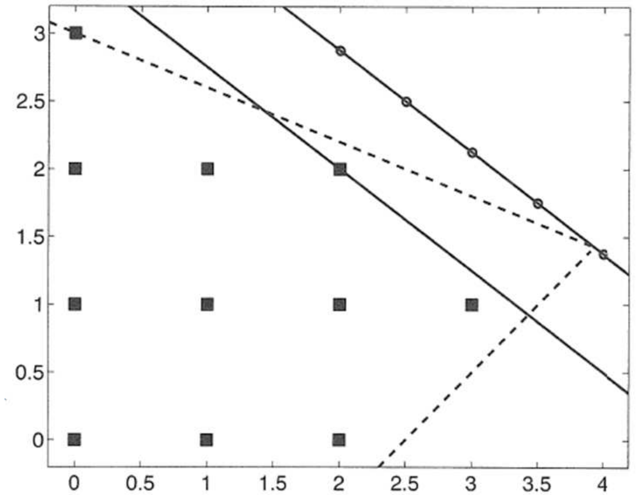
- We first solve the problem graphically. The constraint set Ω for the associated LP problem can be found by calculating the extreme points:

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \quad \boldsymbol{x}^{(2)} = \begin{bmatrix} \frac{5}{2} & 0 \end{bmatrix}^T \quad \boldsymbol{x}^{(3)} = \begin{bmatrix} 0 & 3 \end{bmatrix}^T \quad \boldsymbol{x}^{(4)} = \begin{bmatrix} \frac{55}{14} & \frac{10}{7} \end{bmatrix}^T$$



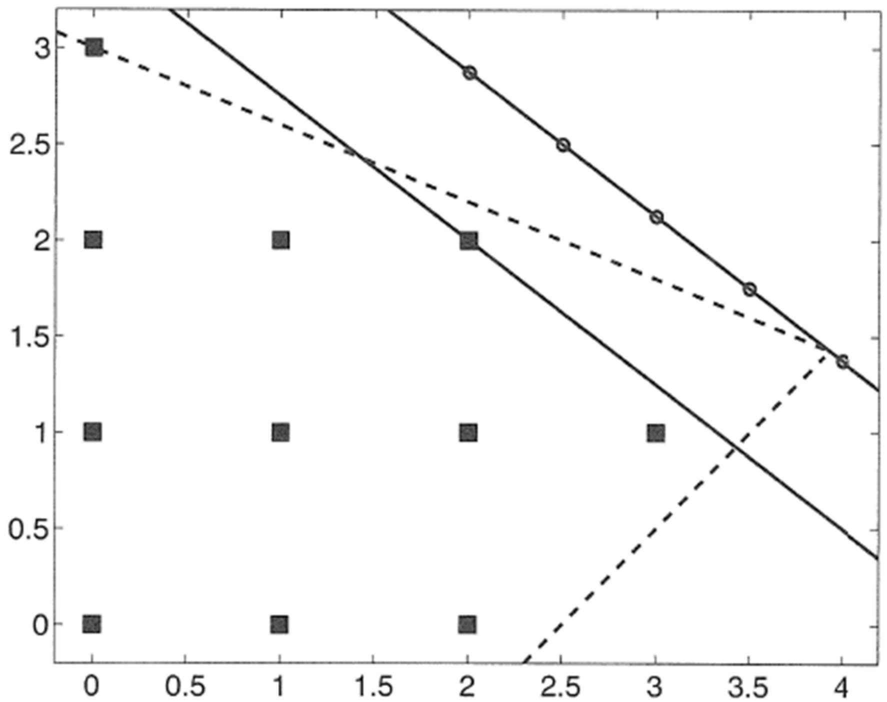
Example

- The solution is obtained by finding the straight line $f = 3x_1 + 4x_2$ with largest f that passes through a feasible point with integer components. This can be accomplished by first drawing the line $f = 3x_1 + 4x_2$ for $f = 0$ and then gradually increasing the values of f, which corresponds to sliding across the feasible region until the straight line passes through the "last" integer feasible point yielding the largest value of the objective function.



Example

- We can see that the optimal solution to the ILP problems is $[2, 2]^T$.



$$\begin{aligned} &\text{maximize } 3x_1 + 4x_2 \\ &\text{subject to } \tfrac{2}{5}x_1 + x_2 \le 3 \qquad x_1, x_2 \ge 0 \\ &\tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 \le 1 \qquad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Example

- We now solve the problem using the Gomory cutting-plane method. First, we represent the associated LP problem in standard form:

$$\begin{aligned} &\text{maximize } 3x_1 + 4x_2 \\ &\text{subject to } \tfrac{2}{5}x_1 + x_2 + x_3 = 3 \\ &\tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 + x_4 = 1 \qquad x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

- Note that we only need the first two components of the solution to be integers. We can start the simplex method because we have an obvious basic feasible solution.

Example

$$\text{maximize } 3x_1 + 4x_2$$
$$\text{subject to } \tfrac{2}{5}x_1 + x_2 + x_3 = 3$$
$$\tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 + x_4 = 1 \qquad x_1, x_2, x_3, x_4 \geq 0$$

- The first tableau is

$$\begin{array}{cccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\ & \frac{2}{5} & 1 & 1 & 0 & 3 \\ & \frac{2}{5} & -\frac{2}{5} & 0 & 1 & 1 \\ \boldsymbol{c}^T & -3 & -4 & 0 & 0 & 0 \end{array}$$

- We bring $\boldsymbol{a}_2$ into the basis and pivot about the element (1,2) to obtain

$$\begin{array}{cccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\ & \frac{2}{5} & 1 & 1 & 0 & 3 \\ & \frac{14}{25} & 0 & \frac{2}{5} & 1 & \frac{11}{5} \\ \boldsymbol{r}^T & -\frac{7}{5} & 0 & 4 & 0 & 12 \end{array}$$

Next, we pivot about the element (2,1) to obtain

$$\begin{array}{cccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\ & 0 & 1 & \frac{10}{14} & -\frac{10}{14} & \frac{20}{14} \\ & 1 & 0 & \frac{10}{14} & \frac{25}{14} & \frac{55}{14} \\ \boldsymbol{r}^T & 0 & 0 & 5 & \frac{5}{2} & \frac{35}{2} \end{array}$$

Example

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{b}$
	0	1	$\frac{10}{14}$	$-\frac{10}{14}$	$\frac{20}{14}$
	1	0	$\frac{10}{14}$	$\frac{25}{14}$	$\frac{55}{14}$
$\boldsymbol{r}^T$	0	0	5	$\frac{5}{2}$	$\frac{35}{2}$

- The corresponding optimal basic feasible solution is $\left[\frac{55}{14} \ \ \frac{10}{7} \ \ 0 \ \ 0\right]^T$, which does not satisfy the integer constraints.
- We start by introducing the Gomory cut corresponding to the first row of the tableau. We obtain

$$-\tfrac{10}{14} - \lfloor -\tfrac{10}{14} \rfloor = -\tfrac{10}{14} - (-1) = \tfrac{4}{14}$$

$$\frac{10}{14}x_3 + \frac{4}{14}x_4 - x_5 = \frac{6}{14}$$

$$\tfrac{20}{14} - \lfloor \tfrac{20}{14} \rfloor = \tfrac{20}{14} - (1) = \tfrac{6}{14}$$

We add this constraint to our tableau:

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
	0	1	$\frac{10}{14}$	$-\frac{10}{14}$	0	$\frac{20}{14}$
	1	0	$\frac{10}{14}$	$\frac{25}{14}$	0	$\frac{55}{14}$
	0	0	$\frac{10}{14}$	$\frac{4}{14}$	-1	$\frac{6}{14}$
$\boldsymbol{r}^T$	0	0	5	$\frac{5}{2}$	0	$\frac{35}{2}$

Example

- Pivoting about the element (3,3) gives

$$\begin{array}{ccccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ & 0 & 1 & 0 & -1 & 1 & 1 \\ & 1 & 0 & 0 & \frac{3}{2} & 1 & \frac{7}{2} \\ & 0 & 0 & 1 & \frac{2}{5} & -\frac{7}{5} & \frac{3}{5} \\ \boldsymbol{r}^T & 0 & 0 & 0 & \frac{1}{2} & 7 & \frac{29}{2} \end{array}$$

The corresponding optimal basic feasible solution is $[7/2, 1, 3/5, 0, 0]^T$, which still does not satisfy the integer constraint.

Example

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
	0	1	0	-1	1	1
	1	0	0	$\frac{3}{2}$	1	$\frac{7}{2}$
	0	0	1	$\frac{2}{5}$	$-\frac{7}{5}$	$\frac{3}{5}$
$\boldsymbol{r}^T$	0	0	0	$\frac{1}{2}$	7	$\frac{29}{2}$

- Next, we construct the Gomory cut for the second row of the tableau

$$\frac{1}{2}x_4 - x_6 = \frac{1}{2}$$

- We add this constraint to our tableau to obtain

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
	0	1	0	-1	1	0	1
	1	0	0	$\frac{3}{2}$	1	0	$\frac{7}{2}$
	0	0	1	$\frac{2}{5}$	$-\frac{7}{5}$	0	$\frac{3}{5}$
	0	0	0	$\frac{1}{2}$	0	-1	$\frac{1}{2}$
$\boldsymbol{r}^T$	0	0	0	$\frac{1}{2}$	7	0	$\frac{29}{2}$

Example

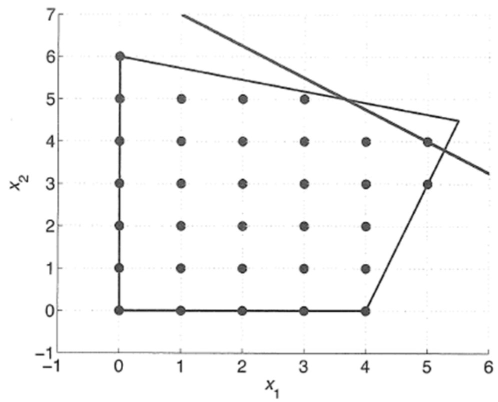
- Pivoting about (4,4), we get

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
	0	1	0	0	1	-2	2
	1	0	0	0	1	3	2
	0	0	1	0	$-\frac{7}{5}$	$\frac{4}{5}$	$\frac{1}{5}$
	0	0	0	1	0	-2	1
$\boldsymbol{r}^T$	0	0	0	0	7	1	14

- In this optimal basic feasible solution, the first two components are integers. Thus, we conclude that the solution to our ILP is $[2, 2]^T$, which agrees with the graphical solution in Figure 19.2.

Example 19.6

- If we are given an ILP problem with inequality constraints as in Example 19.5 but with only integer values in constraint data, then the slack variables and those introduced by the Gomory cuts are automatically integer valued.
- Consider the following ILP problem



$$\text{maximize } 3x_1 + 4x_2$$
$$\text{subject to } 3x_1 - x_2 \leq 12 \qquad x_1, x_2 \geq 0$$
$$3x_1 + 11x_2 \leq 66 \qquad x_1, x_2 \in \mathbb{Z}$$

A graphical solution is shown in Figure 19.3. As in Example 19.5, the solution is obtained by finding the straight line $f = 3x_1 + 4x_2$ with largest f that passes through a feasible point with integer components. This point is $[5, 4]^T$

$$\begin{aligned} &\text{maximize } 3x_1 + 4x_2 \\ &\text{subject to } 3x_1 - x_2 \le 12 \qquad x_1, x_2 \ge 0 \\ &\qquad 3x_1 + 11x_2 \le 66 \qquad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Example 19.6

- We now solve the ILP problem above using the simplex method with Gomory cuts. We first represent the associated LP problem in standard form by introducing slack variables x_3 and x_4. The initial tableau has the form

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{b}$
	3	−1	1	0	12
	3	11	0	1	66
$\boldsymbol{c}^T$	−3	−4	0	0	0

- In this case there is an obvious initial basic feasible solution available, which allows us to initialize the simplex method to solve the problem.

Example 19.6

- After two iterations of the simplex algorithm, the final tableau is

$$\begin{array}{cccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\ & 1 & 0 & \frac{11}{36} & \frac{1}{36} & \frac{11}{2} \\ & 0 & 1 & -\frac{1}{12} & \frac{1}{12} & \frac{9}{2} \\ \boldsymbol{r}^T & 0 & 0 & \frac{7}{12} & \frac{5}{12} & \frac{69}{2} \end{array}$$

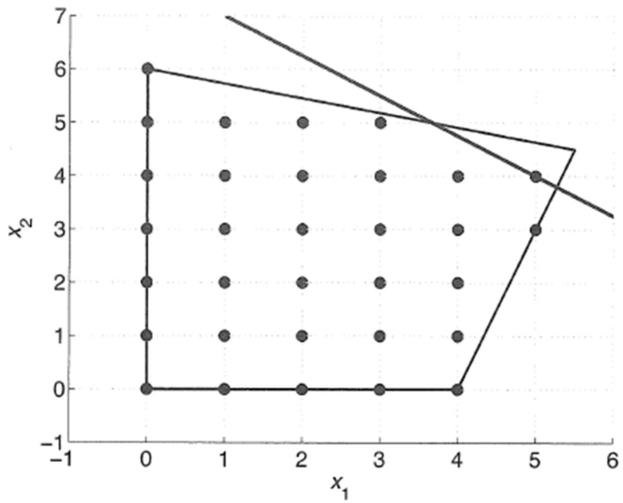
with optimal solution $\boldsymbol{x}^* = \left[\frac{11}{2}\ \frac{9}{2}\ 0\ 0\right]^T$

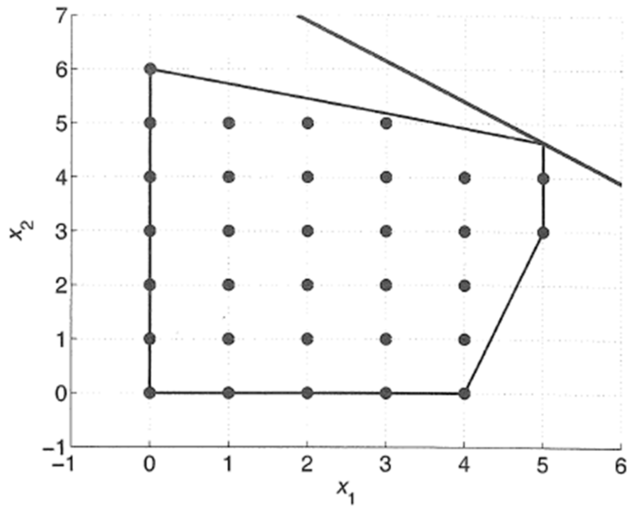
- Both basic components are noninteger. Let us construct a Gomory cut for the first basic component $x_1^* = 11/2$. From the first row of the tableau, the associated constraint equation is

$$x_1 + \frac{11}{36}x_3 + \frac{1}{36}x_4 = \frac{11}{2}$$

Example 19.6

- If we apply the floor operator to this equation as explained before, we get an inequality constraint $x_1 \leq 5$
- A graphical solution of the above problem after adding this inequality constraint to the original LP problem is shown in Figure 19.4. We can see that in this new problem, the first component of the optimal solution is an integer, but not the second. This means that a single Gomory cut will not suffice.





Example 19.6

- To continue with the Gomory procedure for the problem using the simplex method, we first write down the Gomory cut
$$\frac{11}{36}x_3 + \frac{1}{36}x_4 - x_5 = \frac{1}{2}$$
We now obtain a new tableau by augmenting the previous tableau with the above constraint

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
	1	0	$\frac{11}{36}$	$\frac{1}{36}$	0	$\frac{11}{2}$
	0	1	$-\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{9}{2}$
	0	0	$\frac{11}{36}$	$\frac{1}{36}$	-1	$\frac{1}{2}$
$\boldsymbol{r}^T$	0	0	$\frac{7}{12}$	$\frac{5}{12}$	0	$\frac{69}{2}$

At this point, there is no obvious basic feasible solution.

Example 19.6

- However, we can easily use the two-phase method. This yields

$$\begin{array}{ccccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ & 1 & 0 & 0 & 0 & 1 & 5 \\ & 0 & 1 & 0 & \frac{1}{11} & -\frac{3}{11} & \frac{51}{11} \\ & 0 & 0 & 1 & \frac{1}{11} & -\frac{36}{11} & \frac{18}{11} \\ \boldsymbol{r}^T & 0 & 0 & 0 & \frac{4}{11} & \frac{21}{11} & \frac{369}{11} \end{array}$$

  which has all nonnegative reduced cost coefficients. Hence, we obtain the optimal basic feasible solution

$$\boldsymbol{x}^* = \begin{bmatrix} 5 & \frac{51}{11} & \frac{18}{11} & 0 & 0 \end{bmatrix}^T$$

- As expected, the second component does not satisfy the integer constraint.

Example 19.6

	a_1	a_2	a_3	a_4	a_5	b
	1	0	0	0	1	5
	0	1	0	$\frac{1}{11}$	$-\frac{3}{11}$	$\frac{51}{11}$
	0	0	1	$\frac{1}{11}$	$-\frac{36}{11}$	$\frac{18}{11}$
r^T	0	0	0	$\frac{4}{11}$	$\frac{21}{11}$	$\frac{369}{11}$

- Next, we write down the Gomory cut for the basic component $x_2^* = 51/11$ using the numbers in the second row of the tableau

$$\frac{1}{11}x_4 + \frac{8}{11}x_5 - x_6 = \frac{7}{11}$$

- Updating the tableau gives

	a_1	a_2	a_3	a_4	a_5	a_6	b
	1	0	0	0	1	0	5
	0	1	0	$\frac{1}{11}$	$-\frac{3}{11}$	0	$\frac{51}{11}$
	0	0	1	$\frac{1}{11}$	$-\frac{36}{11}$	0	$\frac{18}{11}$
	0	0	0	$\frac{1}{11}$	$\frac{8}{11}$	-1	$\frac{7}{11}$
r^T	0	0	0	$\frac{4}{11}$	$\frac{21}{11}$	0	$\frac{369}{11}$

Again, there is no obvious basic feasible solution.

Example 19.6

- Applying the two-phase method gives

$$\begin{array}{cccccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 & \boldsymbol{b} \\ & 1 & 0 & 0 & -\frac{1}{8} & 0 & \frac{11}{8} & \frac{33}{8} \\ & 0 & 1 & 0 & \frac{1}{8} & 0 & -\frac{3}{8} & \frac{39}{8} \\ & 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{9}{2} & \frac{9}{2} \\ & 0 & 0 & 0 & \frac{1}{8} & 1 & -\frac{11}{8} & \frac{7}{8} \\ \boldsymbol{r}^T & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{21}{8} & \frac{255}{8} \end{array}$$

The corresponding optimal basic feasible solution still does not satisfy the integer constraints; neither the first nor the second components are integer.

Example 19.6

- Next, we introduce the Gomory cut using the numbers in the second row of the previous tableau to obtain

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{a}_7$	$\boldsymbol{b}$
	1	0	0	$-\frac{1}{8}$	0	$\frac{11}{8}$	0	$\frac{33}{8}$
	0	1	0	$\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{39}{8}$
	0	0	1	$\frac{1}{2}$	0	$-\frac{9}{2}$	0	$\frac{9}{2}$
	0	0	0	$\frac{1}{8}$	1	$-\frac{11}{8}$	0	$\frac{7}{8}$
	0	0	0	$\frac{1}{8}$	0	$\frac{5}{8}$	-1	$\frac{7}{8}$
$\boldsymbol{r}^T$	0	0	0	$\frac{1}{8}$	0	$\frac{21}{8}$	0	$\frac{255}{8}$

Example 19.6

- Applying the two-phase method again gives

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{a}_7$	$\boldsymbol{b}$
	1	0	0	0	1	0	0	5
	0	1	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	4
	0	0	1	0	$-\frac{7}{2}$	0	$\frac{1}{2}$	1
	0	0	0	1	$\frac{5}{2}$	0	$-\frac{11}{2}$	7
	0	0	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0
$\boldsymbol{r}^T$	0	0	0	0	1	0	2	31

Note that this basic feasible solution is degenerate – the corresponding basis is not unique. The corresponding optimal basic feasible solution is $[5\ 4\ 1\ 7\ 0\ 0\ 0]^T$ which satisfies the integer constraints.

Example 19.6

- Form this, we see that the integer optimal solution to the original ILP problem is $[5, 4]^T$, which agrees with our graphical solution in Figure 19.3.
- In this example, we note that the final solution to LP problem after introducing slack variables and using the Gomory cutting-plane method is an integer vector. The reason for this, in contrast with Example 19.5, is that the original ILP inequality constraint data has only integers.

Mixed Integer Linear Programming

- A linear programming problem in which not all of the components are required to be integers is called a mixed integer linear programming (MILP) problem. Gomory cuts are also relevant to solving MILP problems. In fact, Example 19.5 is an instance of an MILP problem, because the slack variables in the standard form of the problem are not constrained to be integers. Moreover, the cutting-plane idea also has been applied to nonsimplex methods and nonlinear programming algorithms.